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SMOOTHNESS AND ESTIMATES OF SEQUENCES IN BANACH SPACES*

BY

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ABSTRACT

An upper bound for the order of smoothness of bump functions in Banach spaces without copy of c_0 is found in terms of lower and upper estimates of their sequences. It is also shown that every C^∞-smooth Banach space with symmetric basis either contains c_0 or is isomorphic to ℓ_{2n} for some integer n.

Introduction

We shall be concerned in this paper with smoothness of Banach spaces, in the sense of the existence of smooth bump functions. Recall that a **bump** function is a non-zero real-valued function with bounded support. The existence of such a bump function with a prescribed order of smoothness is of special interest from the point of view of the geometry of Banach spaces, and it is also relevant in different problems of non linear analysis (see [DGZ] for an extensive treatment).

The order of smoothness of L^p $(1 < p < \infty)$ was obtained in [BF], and the same problem was studied in [MT_1], [MT_2] and [Ma] for Orlicz sequence spaces. The connection between modulus of convexity and order of smoothness in uniformly convex spaces was considered in [FGWZ].

In [De_1], [De_2], Deville studied the geometrical structure of very smooth Banach spaces. In particular it is shown in [De_1] that if X is a Banach space without

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copy of c_0 and X is not of exact cotype $2n$ (for some integer n) then there is no bump function with order of smoothness better than cotype, that is, cotype gives an upper bound for the smoothness of X. Here we shall improve this upper bound by introducing an index $l(X)$ that is related to some properties of summability of sequences in X. This will be done in Section 1. It is also shown in [De$_1$] that if X is a C^∞-smooth Banach space without copy of c_0 then X contains ℓ_{2n} for some integer n. We shall prove in Section 2 that if X, in addition, has a subsymmetric basis then X is in fact isomorphic to ℓ_{2n}.

Some of the techniques used here have been introduced by [Ku] and [BF] and developed in [Me], [FWZ], [FGWZ], [De$_1$] and [De$_2$]. The theory of spreading models (see [BL]) will also be useful for us.

A main tool will be the Taylor polynomial of a bump function and in this sense we shall discuss the behaviour of polynomials against sequences with lower and upper estimates. Recall that a map P from a real Banach space X into $\mathbb{R}$ is said to be an N-homogeneous polynomial if there is an N-linear continuous form A on X such that $A(x,\dots,x) = P(x)$ for every $x \in$X. A polynomial of degree (at most) N on X is a finite sum of j-homogeneous polynomials, where $1 \leq j \leq N$. We denote by $\mathcal{P}_N(X)$ the Banach space of all polynomials of degree at most N endowed with the usual norm:

$$\| P \| = \| P_0 \| + \cdots + \| P_N \|,$$

if $P = P_1 + \cdots + P_N$ and P_j is a j-homogeneous polynomial ($1 \leq j \leq N$). For $N \in \mathbb{N}$, let $C^N(X)$ denote the space of all N-times continuously differentiable real functions on X. If $x \in X$ and $f \in C^N(X)$, let $f^{N)}(x)$ denote the N-th derivative of f at x considered as an N-homogeneous polynomial on X. Given $p > 1$, let N be the greatest integer strictly less than p; according to [Me] a function f from X into $\mathbb{R}$ is said to be **H^p-smooth** if $f \in C^N(X)$ and for each $x \in X$ there exist $\delta > 0$ and $M > 0$ such that

$$\| f^{N)}(y) - f^{N)}(z) \| \leq M \| y - z \|^{p-N}, \quad \text{if } \| y - x \| \leq \delta, \| z - x \| \leq \delta.$$

In the same way f is said to be **uniformly H^p-smooth** (UH^p-smooth) if $f \in C^N(X)$ and

$$\| f^{N)}(x) - f^{N)}(y) \| \leq M \| x - y \|^{p-N} \qquad \text{for all } x, y \in X.$$

The Banach space X is said to be **$\mathbf{H}^p$-smooth** (UH^p-smooth, C^n-smooth or C^∞-smooth) if there exists an H^p-smooth (respectively UH$^q, C^n, C^\infty$-smooth) bump function on X.

1. Smoothness and estimates

Let X be a real Banach space and let $1 < p, q < \infty$. A sequence $\{x_n\}$ in X is said to admit an **upper p-estimate** (respectively a **lower q-estimate**) if there exists a constant $M > 0$ such that for every $n \in \mathbb{N}$ and every $a_1, \ldots, a_n \in \mathbb{R}$,

$$\| \sum_{i=1}^{n} a_i x_i \| \leq M(\sum_{i=1}^{n} | a_i |^p)^{1/p}$$

$$(\text{respectively } M(\sum_{i=1}^{n} | a_i |^p)^{1/p} \leq \| \sum_{i=1}^{n} a_i x_i \|).$$

A Banach space X has **property $\mathbf{S_p}$** [KO] (respectively **property $\mathbf{T_q}$** [GJ]) if every weakly null normalized sequence in X has a subsequence that admits an upper p-estimate (respectively lower q-estimate). Reflexive spaces with property S_p have been studied in [CS$_1$], [CS$_2$].

The lower and upper indexes of a Banach space X are defined as follows:

$$l(X) = \sup\{p \geq 1\colon X \text{ has property } S_p\} \in [1, \infty],$$
$$u(X) = \inf\{q \leq \infty\colon X \text{ has property } T_q\} \in [1, \infty].$$

We refer to [GJ] for a detailed study of these indexes. We point out that if X is not a Schur space then

$$l(X) \leq u(X) \leq \inf\{q \geq 1\colon X \text{ has cotype } q\}.$$

On the other hand, for simple examples such as $X = \ell_r \oplus \ell_s$ $(1 < r < s < \infty)$ we have that $r = l(X) < u(X) = s$.

This section will be devoted to prove the following result

MAIN THEOREM: *Suppose that X is a Banach space that does not contain any isomorphic copy of c_0 and $l(X) < \infty$ (respectively $u(X) < \infty$). Then:*

(i) *If $l(X)$ (respectively $u(X)$) is not an even integer then X is not H^q-smooth for $q > l(X)$ (respectively for $q > u(X)$).*

(ii) *If $l(X)$ is an even integer $2n$ (respectively $u(X) = 2m$) and X is H^q-smooth for some $q > 2n$ (respectively for $q > 2m$) then X contains an isomorphic copy of ℓ_{2n} (respectively ℓ_{2m}).*

The proof of this theorem is divided into several partial results and we present all the cases separately.

Throughout this paper a sequence $\{x_n\}$ in X is said to be $\mathcal{P}_N$**-null** if for every M-homogeneous polynomial P on X, where $M \le N$ we have that $\{P(x_n)\}$ is convergent to zero.

PROPOSITION 1.1: *Let X be a UH^q-smooth space and N the greatest integer strictly less than q. Every normalized $\mathcal{P}_N$-null sequence in X has a subsequence with an upper q-estimate.*

Proof: Let f be a UH^q-smooth function such that $f(0) = 0$ and $f(x) = 2$ if $\| x \| \ge 1$. There is a constant $M > 0$ such that if $x, h \in X$

$$\| f^{N)}(x+h) - f^{N)}(x) \| \le M \| h \|^{q-N} .$$

Let $P_N(x)$ denote the Taylor polynomial of degree N of f at x, that is, $P_N(x) = f'(x) + \frac{f^{2)}(x)}{2!} + \cdots + \frac{f^{N)}(x)}{N!}$.

We shall construct a subsequence of $\{x_n\}$ by using an argument of induction. Let

$$\mathcal{Q}_0 = \{P_N(0)(a_1 \cdot): \ | a_1 | \le 1\}.$$

Then $\mathcal{Q}_0$ is a compact set in $\mathcal{P}_N(X)$ and $Q(0) = 0$ for every $Q \in \mathcal{Q}_0$. Since $\{x_n\}$ is a $\mathcal{P}_N$-null sequence in X there exists $n_1 \in \mathbb{N}$ such that $| Q(x_{n_1}) | < 1/2$ for every $Q \in \mathcal{Q}_0$.

Once $n_1 < \cdots < n_k$ have been constructed, let

$$\mathcal{Q}_k = \Big\{ P_N(\sum_{i=1}^{k} a_i x_{n_i})(a_{k+1} \cdot): \ | a_i | \le 1, i = 1, \ldots, k+1 \Big\}.$$

Again, $\mathcal{Q}_k$ is a compact set in $\mathcal{P}_N(X)$ and we may choose an integer $n_{k+1} > n_k$ such that $| Q(x_{n_{k+1}}) | < 1/2^{k+1}$ for every $Q \in \mathcal{Q}_k$.

The subsequence $\{x_{n_k}\}$ constructed in this way satisfies

$$| P_N(\sum_{i=1}^{k-1} a_i x_{n_i})(a_k x_{n_k}) | < 1/2^k$$

for all $a_1, \dots, a_k \in \mathbb{R}$. Therefore, if $| a_i | \leq 1$ for $i = 1, \dots, k$:

$$\begin{aligned} f(\sum_{i=1}^{k} a_i x_{n_i}) = | f(\sum_{i=1}^{k} a_i x_{n_i}) - f(0) | &\leq \sum_{j=1}^{k} | f(\sum_{i=1}^{j} a_i x_{n_i}) - f(\sum_{i=1}^{j-1} a_i x_{n_i}) | \\ &\leq \sum_{j=1}^{k} | f(\sum_{i=1}^{j} a_i x_{n_i}) - f(\sum_{i=1}^{j-1} a_i x_{n_i}) - P_N(\sum_{i=1}^{j-1} a_i x_{n_i})(a_j x_{n_j}) | \\ &+ \sum_{j=1}^{k} | P_N(\sum_{i=1}^{j-1} a_i x_{n_i})(a_j x_{n_j}) | < M(\sum_{j=1}^{k} | a_j |^q) + 1. \end{aligned}$$

Then if $a_1, \dots, a_k \in \mathbb{R}$ and $(\sum_{j=1}^{k} | a_j |^q)^{1/q} = (1/M)^{1/q}$ we have that

$$f\Big(\sum_{i=1}^{k} a_i x_{n_i}\Big) < 2,$$

and therefore $\| \sum_{i=1}^{k} a_i x_{n_i} \| \leq 1$.

Now, if $a_1, \dots, a_k \in \mathbb{R}$ and $a = (\sum_{j=1}^{k} | a_j |^q)^{1/q} > 0$ we have that

$$\| \sum_{i=1}^{k} \frac{a_i}{a M^{1/q}} x_{n_i} \| \leq 1$$

and

$$\| \sum_{i=1}^{k} a_i x_{n_i} \| \leq M^{1/q} (\sum_{j=1}^{k} | a_j |^q)^{1/q}.$$

Then the sequence $\{x_{n_k}\}$ admits an upper q-estimate. ∎

We shall need the following result about weak sequential continuity of polynomials.

PROPOSITION 1.2 ([GJ]): *Let X be a Banach space. If $N < l(X)$ then every N-homogeneous polynomial from X into $\mathbb{R}$ is weakly sequentially continuous.*

LEMMA 1.3: *Let X be a UH^q-smooth space and N the greatest integer strictly less than q. If $N < l(X)$ then X has property S_q.*

Proof: Let $\{x_n\}$ be a weakly null normalized sequence. By Proposition 1.2, $\{x_n\}$ is $\mathcal{P}_N$-null and from Proposition 1.1 it has a subsequence with an upper q-estimate. Therefore X has property S_q. ∎

THEOREM 1.4 ([F], [DGZ, V.3.1.]): *Let X be a Banach space that does not contain any isomorphic copy of c_0. Then X is H^q-smooth if and only if X is UH^q-smooth.*

We now prove the main result for the case where $l(X)$ is not an integer.

THEOREM 1.5: *Let X be a Banach space that does not contain any isomorphic copy o c_0. If $l(X)$ is not an integer then X is not H^q-smooth for $q > l(X)$.*

Proof: Suppose that X is H^q-smooth. Since X does not contain c_0, X is UH^q-smooth by Theorem 1.4. Consider N the greatest integer strictly less than $l(X)$ and assume that $N < q < N+1$. By using Lemma 1.3 we obtain that X would have property S_q and since $l(X) < q$ this is impossible. ∎

Now we study the case where $l(X)$ is an even integer.

LEMMA 1.6: *Let X be a Banach space and let $\{x_n\}$ be a normalized sequence that is $\mathcal{P}_{N-1}$-null but is not $\mathcal{P}_N$-null, where N is an even integer. Then $\{x_n\}$ has a subsequence with a lower N-estimate.*

Proof: Without loss of generality we may assume by the Bessaga–Pelczynski selection principle that $\{x_n\}$ is basic and on the other hand that there exists an N-homogeneous polynomial P such that $P(x_n) \geq \alpha > 0$ for all n.

Let x be fixed and consider the decomposition of P in the following way

$$P(x+h) = P(x) + C(x;h) + P(h)$$

where $C(x;\cdot)$ is a polynomial of degree strictly less than N and $C(x;0) = 0$.

By using an argument of induction we shall construct a subsequence $\{x_{n_k}\}$ such that:

$$\Big|C\Big(\sum_{i=1}^{k} a_i x_{n_i}; a_{k+1}x_{n_{k+1}}\Big)\Big| < 1/2^{k+1} \quad \text{if } |a_i| \leq 1, \quad \text{for } i = 1, \ldots, k+1.$$

In order to obtain this, at each step we consider the following compact set of polynomials of degree at most $N-1$:

$$\mathcal{G}_k = \Big\{C\Big(\sum_{i=1}^{k} a_i x_{n_i}; a_{k+1}\,\cdot\Big) : |a_i| \leq 1, i = 1, \ldots, k+1\Big\}$$

and taking into account that $\{x_n\}$ is $\mathcal{P}_{N-1}$-null we may choose $n_{k+1} > n_k$ such that

$$|Q(x_{n_{k+1}})| < 1/2^{k+1} \quad \text{for every } Q \in \mathcal{G}_k.$$

Then, if $a_1, \dots, a_k \in \mathbb{R}$ and $| a_i | \leq 1$ for $i = 1, \dots, k$:

$$P(\sum_{i=1}^{k} a_i x_{n_i}) = \sum_{i=1}^{k} a_i^N P(x_{n_i}) + \sum_{i=1}^{k-1} C(\sum_{j=1}^{i} a_j x_{n_j})(a_{i+1} x_{n_{i+1}}),$$
$$\alpha \sum_{i=1}^{k} a_i^N - \sum_{i=1}^{k-1} 1/2^{i+1} \geq \alpha \sum_{i=1}^{k} a_i^N - 1.$$

Therefore

$$\alpha(\sum_{i=1}^{k} a_i^N) \leq P(\sum_{i=1}^{k} a_i x_{n_i}) + 1 \leq \| P \| \cdot \| \sum_{i=1}^{k} a_i x_{n_i} \|^N + 1.$$

Let $C > 1$ be the basic constant of $\{x_n\}$. If $a_1, \dots, a_k \in \mathbb{R}$ and $a = \| \sum_{i=1}^{k} a_i x_{n_i} \| > 0$ then $| a_i | \leq 2Ca$ for all $i = 1, \dots, k$. Now,

$$\alpha \sum_{i=1}^{k} (\frac{a_i}{2Ca})^N \leq \| P \| \cdot \| \sum_{i=1}^{k} \frac{a_i}{2Ca} x_{n_i} \| + 1 \leq \| P \| + 1$$

and

$$\alpha \sum_{i=1}^{k} a_i^N \leq (\| P \| + 1) 2^N C^N \| \sum_{i=1}^{k} a_i x_{n_i} \|^N$$

Then the sequence $\{x_{n_k}\}$ admits a lower N-estimate. ∎

THEOREM 1.7: *Let X be a Banach space that does not contain any isomorphic copy of c_0 and suppose that $l(X) = 2n$ is an even integer. If X is H^q-smooth for some $q > 2n$, then X contains an isomorphic copy of ℓ_{2n}.*

Proof: We may suppose that the greatest integer less than q is $2n$. First note that by Proposition 1.1 and Theorem 1.4, X has property S_{2n} and from Proposition 1.2 we have that every weakly null sequence in X is $\mathcal{P}_{2n-1}$-null. Two cases may be given:

FIRST CASE: Every weakly null normalized basic sequence is $\mathcal{P}_{2n}$-null. By Lemma 1.3 we conclude that X has property S_q and this is impossible.

SECOND CASE: There exists a weakly null normalized basic sequence $\{x_k\}$ that is $\mathcal{P}_{2n-1}$-null and is not $\mathcal{P}_{2n}$-null. Without loss of generality we may suppose that $\{x_k\}$ admits an upper $2n$-estimate. By Lemma 1.6 we have that $\{x_k\}$ admits a lower $2n$-estimate and therefore it is equivalent to the unit vector basis in ℓ_{2n}. ∎

In order to get the main theorem for the case where $l(X)$ is an odd integer we need some results about spreading models and polynomials. For the following see for instance [BL]:

Let $\{y_n\}$ be a weakly null normalized basic sequence in a real Banach space X. Then there exists a subsequence $\{x_n\}$ such that for every $k \in \mathbb{N}$ and every $a_1, \ldots, a_k \in \mathbb{R}$ there exists

$$\lim_{n_1 < \cdots < n_k} \| \sum_{i=1}^{k} a_i x_{n_i} \|$$

Let c_{00} be the space of all sequences which are eventually zero and e_n the n^{th} unit vector of the canonical basis for c_{00}. Then

$$L\Big(\sum_{i=1}^{k} a_i e_i\Big) = \lim_{n_1 < \cdots < n_k} \| \sum_{i=1}^{k} a_i x_{n_i} \|$$

for every $a_1, \ldots, a_k \in \mathbb{R}$ and $k \in \mathbb{N}$, defines a norm on c_{00}. The completion of c_{00} with respect to the norm L is denoted by F and is called **spreading model** associated to the sequence $\{x_n\}$. The sequence $\{e_n\}$ is an unconditional basis in F called **fundamental sequence** of the spreading. In addition F has the following properties:

(i) For every $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $\eta \in \mathbb{N}$ such that if $\eta < n_1 < \cdots < n_k$

$$\Big| L\Big(\sum_{i=1}^{k} a_i e_i\Big) - \| \sum_{i=1}^{k} a_i x_{n_i} \| \Big| < \varepsilon$$

for every $a_1, \ldots, a_k \in \mathbb{R}$.

(ii) The sequence $\{e_n\}$ is invariant under spreading with respect to the norm L, that is,

$$L\Big(\sum_{i=1}^{k} a_i e_i\Big) = L\Big(\sum_{i=1}^{k} a_i e_{n_i}\Big)$$

if $n_1 < \cdots < n_k$ and $a_1, \ldots, a_k \in \mathbb{R}$.

THEOREM 1.8: *Let X be a Banach space, let $\{y_n\}$ be a weakly null normalized sequence in X, and let P be an N-homogeneous polynomial on X. Then there exists a subsequence $\{x_n\}$ that admits a spreading model F with fundamental* *sequence $\{e_n\}$, and there exists an N-homogeneous polynomial $\mathbb{P}$ on F such that for all $a_1, \ldots, a_k \in \mathbb{R}$*

$$\mathbb{P}(\sum_{i=1}^{k} a_i e_i) = \lim_{n_1 < \cdots < n_k} P(\sum_{i=1}^{k} a_i x_{n_i}).$$

Proof: Consider a subsequence $\{x_n\}$ of $\{y_n\}$ that admits a spreading model F with fundamental sequence $\{e_n\}$. We shall define $\mathbb{P}$ using Ramsey theorem. Let $k \in \mathbb{N}$ and consider $\mathcal{P}^k(\mathbb{N})$ the set of k-uples of different integers. Let $a_1, \ldots, a_k \in \mathbb{R}$ be fixed and define ψ from $\mathcal{P}^k(\mathbb{N})$ into $\mathbb{R}$ by

$$\psi(\bar{n}) = P(\sum_{i=1}^{k} a_i x_{n_i}), \quad \text{if } \bar{n} = (n_1, \ldots, n_k), \quad n_1 < \cdots < n_k.$$

The function ψ is bounded in $\mathcal{P}^k(\mathbb{N})$, since for every $\bar{n} \in \mathcal{P}^k(\mathbb{N})$

$$| \psi(\bar{n}) | \leq | P(\sum_{i=1}^{k} a_i x_{n_i}) | \leq \| P \| \cdot \| \sum_{i=1}^{k} a_i x_{n_i} \|^N \leq \| P \| (\sum_{i=1}^{k} | a_i |)^N = C.$$

Consider

$$K_1 = \{\bar{n} \in \mathcal{P}^k(\mathbb{N}) \colon 0 \leq \psi(\bar{n}) \leq C\},$$
$$L_1 = \{\bar{n} \in \mathcal{P}^k(\mathbb{N}) \colon -C \leq \psi(\bar{n}) < 0\}.$$

By Ramsey theorem (see e.g. [BL]) there exists an increasing sequence of positive integers $\{\alpha_n^{(1)}\}$ such that if $\bar{n}$ is formed with elements in $\{\alpha_n^{(1)}\}$ then $\psi(\bar{n})$ is in one of the subintervals $[-C, 0]$ or $[0, C]$. Suppose for instance in $[0, C]$.

Consider now the partition of $\mathcal{P}^k(\{\alpha_n^{(1)}\})$

$$K_2 = \{\bar{n} \in \mathcal{P}^k(\{\alpha_n^{(1)}\}) \colon 0 \leq \psi(\bar{n}) \leq C/2\},$$
$$L_2 = \{\bar{n} \in \mathcal{P}^k(\{\alpha_n^{(1)}\}) \colon C/2 < \psi(\bar{n}) \leq C\}.$$

Again by Ramsey theorem there exists a subsequence $\{\alpha_n^{(2)}\}$ of $\{\alpha_n^{(1)}\}$ such that for all $\bar{n}$ formed with elements of $\{\alpha_n^{(2)}\}$ we have that, for instance, $\psi(\bar{n}) \in [0, C/2]$.

In an iterative way we obtain subsequences $\{\alpha_n^{(k)}\}_n$ for every $k \in \mathbb{N}$. Then, we consider the diagonal sequence $\{\alpha_n^{(n)}\}$ and denote $A = \{\alpha_n^{(n)}\}$. By construction there exists

$$\mathbb{P}(\sum_{i=1}^{k} a_i e_i) = \lim_{n_1 < \cdots < n_k; n_i \in A} P(\sum_{i=1}^{k} a_i x_{n_i}).$$

By a diagonalization procedure we may obtain that there exists an infinite subset B of $\mathbb{N}$ such that for every $a_1, \dots, a_k \in \mathbb{Q}$, there exists

$$\mathbb{P}(\sum_{i=1}^{k} a_i e_i) = \lim_{n_1 < \cdots < n_k; n_i \in B} P(\sum_{i=1}^{k} a_i x_{n_i}).$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and P is uniformly continuous on bounded sets of X it follows that there exists

$$\mathbb{P}(\sum_{i=1}^{k} a_i e_i) = \lim_{n_1 < \cdots < n_k; n_i \in B} P(\sum_{i=1}^{k} a_i x_{n_i})$$

for every $k \in \mathbb{N}$ and every $a_1, \dots, a_k \in \mathbb{R}$. Without loss of generality we may consider that $B = \mathbb{N}$ taking a subsequence of $\{x_n\}$ that will have F as spreading model too. Now, define

$$\mathbb{P}(\sum_{i=1}^{k} a_i e_i) = \lim_{n_1 < \cdots < n_k} P(\sum_{i=1}^{k} a_i x_{n_i}).$$

Then $\mathbb{P}$ is defined on the space E of all finite linear combinations of $\{e_n\}$ and by construction

$$\mathbb{P}(\sum_{i=1}^{k} a_i e_i) \leq \| P \| \cdot L(\sum_{i=1}^{k} a_i e_i)^N$$

for every $k \in \mathbb{N}$ and every $a_1, \dots, a_k \in \mathbb{R}$. In order to prove that $\mathbb{P}$ is an N-homogeneous polynomial on E by [BS] we only need to check the following:

(i) For every fixed $a, b \in E$ the function $\mathbb{P}(a + tb)$ is a polynomial in t of degree at most N. Indeed, choose $m \in \mathbb{N}$ such that $a = \sum_{i=1}^{m} a_i x_i$ and $b = \sum_{i=1}^{m} b_i x_i$; then

$$\mathbb{P}(a + tb) = \lim_{n_1 < \cdots < n_m} P_{n_1, \dots, n_m}(t)$$

where $P_{n_1, \dots, n_m}(t) = P(\sum_{i=1}^{m} (a_i + tb_i) x_{n_i})$ is a polynomial of degree at most N in $\mathbb{R}$. Then $\mathbb{P}(a + tb)$ is a pointwise limit of polynomials in $\mathbb{R}$ of degree at most N.

(ii) $\mathbb{P}(ta) = t^N \mathbb{P}(a)$, and this is clear.

Since E is dense in F and $\mathbb{P}$ is uniformly continuous on bounded sets of E we can extend $\mathbb{P}$ to an N-homogeneous polynomial from F into $\mathbb{R}$ and by continuity we have

$$| \mathbb{P}(x) | \leq \| P \| \cdot \| x \|^N \qquad \text{for every } x \in F. \qquad \blacksquare$$

PROPOSITION 1.9: *Let X be a Banach space with unconditional basis $\{x_n\}$. Suppose that there exists an N-homogeneous polynomial P on X such that $P(x_n) \geq \alpha > 0$ for every $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ admits a lower N-estimate.*

Proof: We consider $\bar{X}$ the complexified space of X, endowed with the norm

$$\| x + iy \| = \sup_{0 \leq \theta \leq 1} \| x \cos\theta + y \sin\theta \|$$

Then $\{x_n\}$ is an unconditional basis in $\bar{X}$ and P admits a unique extension to a polynomial $\bar{P}$ from $\bar{X}$ into $\mathbb{C}$, and there exists a N-linear continuous map A from $\bar{X} \times \cdots \times \bar{X}$ into $\mathbb{C}$ such that $\bar{P}(x) = A(x, \ldots, x)$ for all $x \in \bar{X}$.

Let $\{s_n(t)\}$ be the Generalized Rademacher Functions introduced in [AG]. These functions are defined as follows: Fix $N \in \mathbb{N}$, and let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_N$ denote the N-roots of unity. Let $s_1\colon [0,1] \longrightarrow \mathbb{C}$ be the step function taking the value α_j on $(\frac{j-1}{N}, \frac{j}{N})$ for $j = 1, 2, \ldots, N$. Then, assuming that s_{n-1} has been defined, define s_n in the usual way; fix any of the N^{n-1} subintervals I of [0,1] used in the definition of s_{n-1}, divide I into N equal intervals $I_1, \ldots, I_N$, and set $s_n(t) = \alpha_j$ if $t \in I_j$. The sequence of functions $\{s_n\}$ satisfies that $| s_n(t) | = 1$ and they are orthogonal in the sense that for every $k_1, \ldots, k_N \in \mathbb{N}$

$$\int_0^1 s_{i_1}(t) \ldots s_{i_N}(t) = \begin{cases} 1, & \text{if } i_1 = \cdots = i_N, \\ 0, & \text{otherwise.} \end{cases}$$

Then, if $a_1, \ldots, a_n \in \mathbb{R}$,

$$\begin{aligned} \alpha \sum_{i=1}^n | a_i |^N \leq \sum_{i=1}^n | a_i |^N P(x_i) = \sum_{i=1}^n \bar{P}(| a_i | x_i) = \int_0^1 \bar{P}(\sum_{i=1}^n | a_i | s_i(t) x_i) dt \\ = \int_0^1 A\Big(\sum_{i=1}^n | a_i | s_i(t) x_i, \ldots, \sum_{i=1}^n | a_i | s_i(t) x_i\Big) dt \\ \leq \int_0^1 \| A \| \cdot \| \sum_{i=1}^n | a_i | s_i(t) x_i \|^N \leq \| A \| C^N \| \sum_{i=1}^n a_i x_i \|^N \end{aligned}$$

where C is the basic constant of $\{x_n\}$. Therefore $\{x_n\}$ admits a lower N-estimate. ∎

LEMMA 1.10: *Let X be a Banach space and let $\{y_n\}$ be a normalized $\mathcal{P}_{N-1}$-null sequence in X, where N is an odd integer. Let $P_1, \ldots, P_s$ be N-homogeneous* *polynomials on X and let $\varepsilon > 0$. Then there exist $\eta \in \mathbb{N}$ and a subsequence $\{x_n\}$ of $\{y_n\}$ such that $| P_i(x_n - x_m) | < \varepsilon$, if $\eta < n < m$ and $i = 1, \ldots, s$.*

Proof: Let $\{x_n\}$ be a subsequence of $\{y_n\}$ that admits a spreading model F with fundamental sequence $\{e_n\}$. Taking a subsequence if necessary, by Theorem 1.8 there exist polynomials $\mathbb{P}_1, \ldots, \mathbb{P}_s$ associated to $P_1, \ldots, P_s$, verifying

$$\mathbb{P}_r(\sum_{j=1}^{k} a_j e_j) = \lim_{n_1 < \cdots < n_k} P_r(\sum_{j=1}^{k} a_j x_{n_j}).$$

Let us denote $\alpha_r = \lim_{n_1} P_r(x_{n_1}) = \mathbb{P}_r(e_n)$ for all $n \in \mathbb{N}$ and each $r = 1, \ldots, s$. We are going to see that

$$\mathbb{P}_r(\sum_{j=1}^{k} a_j x_j) = \alpha_r \sum_{j=1}^{k} a_j^N, \tag{1}$$

for every $r = 1, \ldots, s$ and every $a_1, \ldots, a_k \in \mathbb{R}$.

Fix $a_1, \ldots, a_k \in \mathbb{R}$. Then

$$P_r(\sum_{j=1}^{k} a_j x_{n_j}) = \sum_{j=1}^{k} a_j^N P_r(x_{n_j}) + \sum_{j=1}^{k-1} C_r(\sum_{i=1}^{j} a_i x_{n_i}; a_{j+1} x_{n_{j+1}}),$$

where $C_r(x; \cdot)$ is a polynomial of degree strictly less than N. By Proposition 1.2, $\{x_n\}$ is a $\mathcal{P}_{N-1}$-null sequence. Using the same procedure as in Proposition 1.1, if $\delta > 0$ and $m \in \mathbb{N}$ are given we can choose $m < n_1 < \cdots < n_k$ such that

$$| C_r(\sum_{i=1}^{j} a_i x_{n_i}; a_{j+1} x_{n_{j+1}}) | < \frac{\delta}{k}. \tag{2}$$

Therefore, since there exists

$$\lim_{n_1 < \cdots < n_k} \left[P_r(\sum_{j=1}^{k} a_j x_{n_j}) - \sum_{j=1}^{k} a_j^N P_r(x_{n_j}) \right] = \lim_{n_1 < \cdots < n_k} P_r(\sum_{j=1}^{k} a_j x_{n_j}) - \alpha_r \sum_{j=1}^{k} a_j^N$$

by (2) this limit has to be necessarily 0, and this proves (1). Then $\mathbb{P}_r(e_1 - e_2) = \alpha_r(1^N + (-1)^N) = 0$ and there exists, by definition of $\mathbb{P}_r$, some $\eta \in \mathbb{N}$ such that if $\eta < n < m$,

$$| P_r(x_n - x_m) | < \varepsilon$$

for every $r = 1, \ldots, s$ as we required. ∎

We need the following result analogous to Lemma 1.6.

LEMMA 1.11: *Let X be a Banach space and let $\{x_n\}$ be a normalized sequence that is P_{N-1}-null but is not P_N-null. Then $\{x_n\}$ has a subsequence with a lower r-estimate for every $r > N$.*

Proof: Taking a subsequence if necessary we may suppose that there exists an N-homogeneous polynomial P on X such that $P(x_n) \geq 1$ for all n and $\{x_n\}$ admits a spreading model F with fundamental sequence $\{e_n\}$. By Theorem 1.8 there exists an N-homogeneus polynomial $\mathbb{P}$ on F such that, in particular, $\mathbb{P}(e_n) \geq 1$ for all n. By Proposition 1.9 the sequence $\{e_n\}$ admits a lower N-estimate in F. The result now follows from [GJ]. ∎

THEOREM 1.12: *Let X be a Banach space without any isomorphic copy of c_0. Suppose that $l(X) = N$ is an odd integer. Then X is not H^p-smooth if $p > N$.*

Proof: We may suppose that $N < p < N + 1$. Since X does not contain c_0, X is UH^p-smooth by Theorem 1.4. On the other hand by Proposition 1.2 every polynomial of degree at most $N - 1$ is weakly sequentially continuous. Two cases may be given:

FIRST CASE: Every polynomial of degree N is weakly sequentially continuous at zero. Then every weakly null sequence in X is $\mathcal{P}_N$-null and by Proposition 1.1 it follows that X has property S_p. This is impossible since $p > N$.

SECOND CASE: There exists an N-homogeneous polynomial on X that is not weakly sequentially continuous at zero. Let $\{x_n\}$ be a weakly null normalized sequence and let P be an N-homogeneous polynomial on X such that $P(x_n) \geq \alpha > 0$ for all n. By Lemma 1.11 we may suppose, taking a subsequence, that $\{x_n\}$ has a lower r-estimate for all $r > N$. Fix q such that $N < q < p$. There is a constant $D > 0$ such that if $a_1, \ldots, a_n \in \mathbb{R}$,

$$(3) \qquad (\sum_{i=1}^{n} | a_i |^q)^{1/q} \leq D \| \sum_{i=1}^{n} a_i x_i \| .$$

Now consider a UH^p-smooth function f from X into $\mathbb{R}$ such that $f(0) = 0$ and $f(x) = 2$ if $\| x \| \geq 1$. There is a constant $C > 0$ such that for all $x, y \in X$

$$\| f^{N)}(x) - f^{N)}(y) \| \leq C \| x - y \|^{p-N} .$$

Denote by $P_k(x)$ the Taylor polynomial of degree k of f at x, and let $Q_N(x) = \frac{f^{N)}(x)}{N!}$.

Next we are going to construct by induction a block basis $\{y_n\}$ of $\{x_n\}$. Let $\varepsilon > 0$ be fixed and $N < q < p$. Then there exists $0 < \alpha < \frac{1}{2}$ such that if $\| h \| \leq 2\alpha$

$$\| f^{N)}(x+h) - f^{N)}(x) \| \leq \varepsilon \| h \|^{q-N} .$$

Let $y_0 = 0$ and consider

$$\mathcal{F}_0 = \{P_{N-1}(0)(\varepsilon_1 \cdot) : \varepsilon_1 = \pm 1\},$$
$$\mathcal{G}_0 = \{Q_N(0)(\varepsilon_1 \cdot) : \varepsilon_1 = \pm 1\}.$$

Since $\mathcal{G}_0$ is a finite set of polynomials of degree N it follows from Lemma 1.10 that there exist an infinite set A_1 of $\mathbb{N}$ and some $\eta \in \mathbb{N}$ such that

$$| Q(\alpha x_{n_1} - \alpha x_{m_1}) | < \frac{1}{4}, \text{ for all } n_1, m_1 \in A_1 \text{ with } \eta < n_1 < m_1, \text{ and all } Q \in \mathcal{G}_0.$$

Since $\mathcal{F}_0$ is a finite set of polynomials of degree at most $N-1$ and $\{\alpha x_n\}$ is $\mathcal{P}_{N-1}$-null we may choose $n_1, m_1 \in A_1$ such that $\eta < n_1 < m_1$ and

$$| P(\alpha x_{n_1} - \alpha x_{m_1}) | < \frac{1}{4}, \quad \text{for all } P \in \mathcal{F}_0.$$

Then, we define $y_1 = \alpha x_{n_1} - \alpha x_{m_1}$. Once we have constructed $y_k = \alpha x_{n_k} - \alpha x_{m_k}$ with $m_{k-1} < n_k < m_k$ for $1 \leq k \leq r$, consider

$$\mathcal{F}_r = \{P_{N-1}(\sum_{i=1}^{r} \varepsilon_i y_i)(\varepsilon_{r+1} \cdot) : \varepsilon_i = \pm 1, i = 1, \ldots, r+1\},$$
$$\mathcal{G}_r = \{Q_N(\sum_{i=1}^{r} \varepsilon_i y_i)(\varepsilon_{r+1} \cdot) : \varepsilon_i = \pm 1, i = 1, \ldots, r+1\}.$$

We may choose $n_{r+1}, m_{r+1} \in \mathbb{N}$, with $m_r < n_{r+1} < m_{r+1}$ such that

$$| Q(\alpha x_{n_{r+1}} - \alpha x_{m_{r+1}}) | < \frac{1}{2^{r+2}} \quad \text{for every } Q \in \mathcal{F}_r \cup \mathcal{G}_r.$$

Now define $y_{r+1} = \alpha x_{n_{r+1}} - \alpha x_{m_{r+1}}$. Then, since $\| y_r \| \leq 2\alpha$ for each r,

$$\left| f(\sum_{i=1}^{r} \varepsilon_i y_i) - f(\sum_{i=1}^{r-1} \varepsilon_i y_i) \right| \leq \left| f(\sum_{i=1}^{r} \varepsilon_i y_i) - f(\sum_{i=1}^{r-1} \varepsilon_i y_i) - P_N(\sum_{i=1}^{r-1} \varepsilon_i y_i)(\varepsilon_r y_r) \right|$$
$$+ \left| P_{N-1}(\sum_{i=1}^{r-1} \varepsilon_i y_i)(\varepsilon_r y_r) \right| + \left| Q_N(\sum_{i=1}^{r-1} \varepsilon_i y_i)(y_r) \right| \leq \varepsilon \| y_r \|^q + 1/2^{r+1} + 1/2^{r+1}.$$

Since $\{x_n\}$ is a basic sequence, we have that $\{\| y_i \|\}$ is not convergent to zero. Since X does not contain c_0, by the results of Bessaga–Pelczynski in [BP] the set

$$K = \Big\{ \sum_{i=1}^{r} \varepsilon_i y_i \colon \varepsilon_i = \pm 1, \quad r \in \mathbb{N} \Big\}$$

is not bounded. Therefore there exist a choice of signs $\varepsilon_i = \pm 1$ and $r \in \mathbb{N}$ such that

$$\| \sum_{i=1}^{r-1} \varepsilon_i y_i \| < 1 \quad \text{and} \quad \| \sum_{i=1}^{r} \varepsilon_i y_i \| \geq 1.$$

Then, using (3),

$$\begin{aligned} \sum_{i=1}^{r-1} \| y_i \|^q &\leq \sum_{i=1}^{r-1} (\alpha + \alpha)^q \leq 2^q \sum_{i=1}^{r-1} (\alpha^q + \alpha^q) \\ &\leq 2^q D^q \| \sum_{i=1}^{r-1} \varepsilon_i (\alpha x_{n_i} - \alpha x_{m_i}) \|^q \leq (2D)^q. \end{aligned}$$

Now,

$$2 = | f(\sum_{i=1}^{r} \varepsilon_i y_i) - f(0) | \leq \varepsilon (2D)^q + \varepsilon\alpha + 1 < \varepsilon((2D)^q + 1) + 1,$$

and by choosing $\varepsilon < \frac{1}{(2D)^q + 1}$, we get a contradiction. ∎

Analogous results can be also obtained for $u(X)$ instead of $l(X)$, as we shall see in the following theorem.

THEOREM 1.13: *Suppose that X is a Banach space that does not contain any isomorphic copy of c_0 and $u(X) < \infty$.*

(i) *If $u(X)$ is not an even integer then X is not H^p-smooth for $p > u(X)$.*
(ii) *If $u(X) = 2n$ is an even integer and X is H^p-smooth for some $p > 2n$ then X contains an isomorphic copy of ℓ_{2n}.*

Proof: Consider N the greatest integer less than $u(X)$ and let $u(X) < p < N+1$. Suppose that X is H^p-smooth; in particular X is not a Schur space and there exist weakly null normalized sequences in X. Two cases may be given:

FIRST CASE: There exists a weakly null normalized sequence $\{x_n\}$ that is $\mathcal{P}_N$-null. Since X is H^p-smooth, by Proposition 1.1,$\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with an upper p-estimate. Let $p > q > u(X)$; since X has property T_q, some subsequence of $\{x_{n_k}\}$ should have a lower q-estimate and this is impossible.

SECOND CASE: No weakly null normalized sequence in X is $\mathcal{P}_N$-null. It follows from Lemma 1.11 that X has property T_r for all $r > N$ and then $u(X) = N$. Now fix $N-1 < r < N$. Since X has not property T_r, there exists a sequence $\{x_n\}$ such that no subsequence of it has a lower r-estimate, and again by Lemma 1.11 $\{x_n\}$ is $\mathcal{P}_{N-1}$-null. In the case that $N = 2n$ is an even integer we obtain from Proposition 1.1 and Lemma 1.11 that $\{x_n\}$ has a subsequence equivalent to the unit vector basis of ℓ_{2n}. In the case that N is an odd integer we proceed as in Theorem 1.12 to obtain a contradiction. ∎

As a consequence of the main theorem we obtain the following

COROLLARY 1.14: *If X is a C^∞-smooth Banach space that does not contain an isomorphic copy of c_0 then $l(X) = 2n$ and $u(X) = 2m$ are even integers. Besides, the following are equivalent:*

(i) *X contains an isomorphic copy of ℓ_{2k}.*
(ii) *There is a subspace Y of X such that $l(Y) = 2k$.*
(iii) *There is a subspace Y of X such that $u(Y) = 2k$.*

In particular X contains a copy of ℓ_{2n} and ℓ_{2m}.

2. C^∞-smooth spaces with subsymmetric basis

In this section we are going to see that the existence of a subsymmetric basis has a strong impact on the structure of C^∞-smooth Banach spaces.

Recall that a basis $\{e_n\}$ of a Banach space X is said to be **subsymmetric** if it is unconditional and, for every increasing sequence $\{n_i\}$ of integers, $\{e_{n_i}\}$ is equivalent to $\{e_n\}$. It is well known (see e.g. [LT$_1$]) that if X has a subsymmetric basis then it admits an equivalent norm for which $\{e_n\}$ is invariant under spreading, that is

$$\left\|\sum_{i=1}^{k} a_i e_i\right\| = \left\|\sum_{i=1}^{k} a_i e_{n_i}\right\|$$

for every $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{R}$. We shall always consider this norm on X.

THEOREM 2.1: *Let X be a Banach space with subsymmetric basis $\{e_k\}$. Assume that X does not contain any isomorphic copy of c_0 and contains an isomorphic copy of ℓ_{2n}. Then X is C^{2n}-smooth if and only if X is isomorphic to ℓ_{2n}.*

Proof: Suppose that X is C^{2n}-smooth. Then X is superreflexive (see, for instance, [DGZ]) and therefore $\{e_k\}$ is weakly null. Now two cases may be given.

FIRST CASE: There is an integer $m, 2 \leq m \leq 2n$, such that $\{e_k\}$ is $\mathcal{P}_{m-1}$-null and is not $\mathcal{P}_m$-null. Combining Propositions 1.1 and 1.9 we obtain that $\{e_k\}$ has a subsequence with both a lower and upper m-estimate. Since $\{e_k\}$ is subsymmetric it follows that X is isomorphic to ℓ_m and therefore $m = 2n$.

SECOND CASE: The sequence $\{e_k\}$ is $\mathcal{P}_{2n}$-null. Since $\{e_k\}$ is subsymmetric, the spreading model built over $\{e_k\}$ coincides with X. Therefore if P is a polynomial of degree $2n$ on X and $\mathbb{P}$ is the associated polynomial given by Theorem 1.8, it is easily seen that

$$\mathbb{P}(\sum_{i=1}^{k} a_i e_i) = \lim_{n_1 < \cdots < n_k} P(\sum_{i=1}^{k} a_i e_{n_i}) = 0 \tag{4}$$

for every $k \in \mathbb{N}$ and every $a_1, \ldots, a_k \in \mathbb{R}$.

By a standard perturbation argument we may choose a normalized block basis $\{x_k\}$ of $\{e_k\}$ that is equivalent to the unit vector basis of ℓ_{2n}. There is a sequence of finite integers blocks $F_1 < F_2 < \cdots$ such that

$$x_i = \sum_{j \in F_i} a_j e_j.$$

Now let f be a C^{2n}-smooth function such that $f(0) = 0$ and $f(x) = 3$ if $\| x \| \geq 1$. Let $\varepsilon > 0$ be fixed and let $P(x)$ denote the Taylor polynomial of degree $2n$ of f at x. For $x \in X$ consider

$$\lambda(x) = \sup\{\delta \leq 1 \colon \| f^{2n)}(x+h) - f^{2n)}(x) \| \leq \varepsilon, \| h \| \leq \delta\}$$

We start with $y_0 = 0$ and let $\alpha_1 = \frac{\lambda(0)}{2}$. By (4) we may choose an injection $\sigma_1 \colon F_1 \to \mathbb{N}$ such that $\sigma_1(j) < \sigma_1(j')$ if $j < j'$ and

$$| P(0)(\alpha_1 \sum_{j \in F_1} a_j e_{\sigma_1(j)}) | < \frac{1}{2}.$$

Then define $y_1 = \alpha_1 \sum_{j \in F_1} a_j e_{\sigma_1(j)}$. Suppose we have constructed $\sigma_i \colon F_i \to \mathbb{N}$ for $i = 1, \ldots, r$ such that $\sigma_i(j) < \sigma_i(j')$ if $j < j'$, $\sigma_1(F_1) < \cdots < \sigma_r(F_r)$ and $y_i = \alpha_i \sum_{j \in F_i} a_j e_{\sigma_i(j)}$ where $\alpha_i = \frac{1}{2}\lambda(y_1 + \cdots + y_{i-1})$. Then we consider $P(y_1 + \cdots + y_r)$ and again by (4) we can choose an injection $\sigma_{r+1} \colon F_{r+1} \to \mathbb{N}$ such that $\sigma_{r+1}(j) < \sigma_{r+1}(j')$ if $j < j'$, $\sigma_r(F_r) < \sigma_{r+1}(F_{r+1})$ and

$$| P(y_1 + \cdots + y_r)(\alpha_{r+1} \sum_{j \in F_{r+1}} a_j e_{\sigma_{r+1}(j)}) | < \frac{1}{2^{r+1}},$$

where $\alpha_{r+1} = \frac{1}{2}\lambda(y_1 + \cdots + y_r)$. Now define

$$y_{r+1} = \alpha_{r+1} \sum_{j \in F_{r+1}} a_j e_{\sigma_{r+1}(j)}.$$

Then

$$\begin{aligned} \mid f(\sum_{i=1}^{r} y_i) - f(\sum_{i=1}^{r-1} y_i) \mid &\leq \mid f(\sum_{i=1}^{r} y_i) - f(\sum_{i=1}^{r-1} y_i) - P(\sum_{i=1}^{r-1} y_i)(y_r) \mid \\ &+ \mid P(\sum_{i=1}^{r-1} y_i)(y_r) \mid \leq \varepsilon \parallel y_r \parallel^{2n} + \frac{1}{2^{r+1}} = \varepsilon \alpha_r^{2n} + \frac{1}{2^{r+1}}. \end{aligned} \tag{5}$$

Since $\{x_n\}$ has a lower $2n$-estimate, there exists $C > 0$ such that

$$\sum_{i=1}^{r} \mid a_i \mid^{2n} \leq C \parallel \sum_{i=1}^{r} a_i x_i \parallel^{2n}$$

and therefore

$$\begin{aligned} \mid f(\sum_{i=0}^{r} y_i) - f(0) \mid &\leq \sum_{i=0}^{r} (\varepsilon \mid \alpha_i \mid^{2n} + \frac{1}{2^{i+1}}) \leq C\varepsilon \parallel \sum_{i=0}^{r} \alpha_i x_i \parallel^{2n} + 1 \\ &= \varepsilon C \parallel \sum_{i=0}^{r} \alpha_i \sum_{j \in F_i} a_j e_j \parallel^{2n} + 1 = \varepsilon \parallel \sum_{i=0}^{r} \alpha_i \sum_{j \in F_i} a_j e_{\sigma_i(j)} \parallel^{2n} + 1. \end{aligned} \tag{6}$$

Suppose that the set $K = \{\sum_{n=1}^{r} y_n : r \in \mathbb{N}\}$ is bounded. Since $\{y_n\}$ is unconditional and X does not contain c_0, by [BP] the series $\sum_{i=1}^{\infty} y_i$ is unconditionally convergent and K is relatively compact. Therefore

$$\lambda(K) = \sup\{\delta \leq 1 : \parallel f^{2n)}(x+h) - f^{2n)}(x) \parallel \leq \varepsilon, \quad \parallel h \parallel \leq \delta, \quad x \in K\} > 0$$

and $\alpha_n \geq \lambda(K)/2 > 0$ for all n. But this is impossible, since $\alpha_n \leq \parallel y_n \parallel$ and the sequence $\{\parallel y_n \parallel\}$ is convergent to 0.

Hence K is not bounded and we may choose $r \in \mathbb{N}$ such that

$$\parallel \sum_{i=1}^{r-1} y_i \parallel \leq 1 \qquad \text{and} \qquad \parallel \sum_{i=1}^{r} y_i \parallel > 1.$$

From (5) and (6) we obtain that

$$\begin{aligned} 3 &= \mid f(\sum_{i=1}^{r} y_i) - f(0) \mid \\ &\leq \mid f(\sum_{i=1}^{r} y_i) - f(\sum_{i=1}^{r-1} y_i) \mid + \mid f(\sum_{i=1}^{r-1} y_i) - f(0) \mid \leq \varepsilon(C+1) + 2. \end{aligned}$$

It follows a contradiction by choosing $\varepsilon < \frac{1}{C+1}$. ∎

By using Corollary 1.14 and Theorem 2.1 we obtain

COROLLARY 2.2: *Let X be a C^∞-smooth Banach space with subsymmetric basis. Then either X contains an isomorphic copy of c_0 or X is isomorphic to ℓ_{2n} for some integer n.*

Remark 2.3: We note that C^∞-smooth Banach spaces containing c_0 and with subsymmetric basis need not be isomorphic to c_0. Consider for instance the Orlicz sequence space h_M where $M(t) = t\exp\{-1/t\}$. As shown in [MT$_2$] the space h_M is C^∞-smooth, has symmetric basis and contains c_0 but is not isomorphic to c_0.

Next we are going to apply our results to some well known classes of Banach spaces with symmetric basis. Recall that a basis is said to be symmetric if it is equivalent to each of its permutations. Every symmetric basis is subsymmetric [LT$_1$].

Let M be an Orlicz function that satisfies Δ_2-property and ℓ_M the Orlicz sequence space associated to M. The lower Boyd index associated to M (see, e.g., [LT$_1$]) is defined by

$$\alpha_M = \sup\left\{p \geq 1\colon \sup\left\{\frac{M(uv)}{u^p M(v)}\colon 0 < u, v \leq 1\right\} < \infty\right\}.$$

For these spaces $l(\ell_M) = \alpha_M$ (see [K]) and it is shown in [Ma] that ℓ_M is H^q-smooth if $q < l(\ell_M)$. The exact order of smoothness of ℓ_M has been computed in [Ma], [MT$_1$] and [MT$_2$]. Some of the results obtained there can be derived as a consequence of Theorem 2.1.

COROLLARY 2.4: *Let ℓ_M be the Orlicz sequence space associated to an Orlicz function M that verifies the Δ_2-property.*

(i) *If $\alpha_M = 2k$ is an even integer and ℓ_M is C^{2k}-smooth then ℓ_M is isomorphic to ℓ_{2k}.*

(ii) *ℓ_M is H^{α_M}-smooth if and only if ℓ_M has property S_{α_M}.*

Proof: (i) Since ℓ_M has symmetric basis and contains ℓ_{2k} [LT$_1$], the result follows from Theorem 2.1.

(ii) By combining the results in [Ma] and [K] we obtain that if ℓ_M has property S_{α_M} then X is H^{α_M}-smooth. The converse follows from Lemma 1.3. ∎

Let $d(w,p)$ be the Lorentz sequence space associated to a real decreasing to zero sequence $w = \{w_n\}$ and $p > 1$ (see [LT$_1$]). Since $d(w,p)$ contains ℓ_p, it is not H^q-smooth if $q > p$ and p is not an even integer. Using again Theorem 2.1 we obtain

COROLLARY 2.5: *The Banach space $d(w, 2k)$ is not C^{2k}-smooth.*

Remark 2.6: Let X be a Banach space without any copy of c_0 and such that $l(X)$ is not an even integer. We have seen in Section 1 that, in this case, $l(X)$ is an upper bound for the order of smoothness of bump functions on X. This bound is sharp for sequence spaces such as $X = \ell_p, X = \ell_p \oplus \ell_q$ or $X = \ell_M$. Indeed, in these cases X is H^r-smooth for $r < l(X)$ and X is not H^r-smooth for $r > l(X)$. On the other hand there are examples where X is not H^r-smooth where $r = l(X)$; consider for instance the Orlicz sequence space ℓ_M associated to the Orlicz function $M(t) = -t^p \log t$ $(1 < p < \infty)$. Here $l(\ell_M) = p$ and using [K] and Lemma 1.3 it follows that ℓ_M is not H^p-smooth.

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